

INNER RADIUS OF NODAL DOMAINS OF QUANTUM ERGODIC EIGENFUNCTIONS

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ABSTRACT. In this short note we show that the lower bounds of Mangoubi [Ma08b] on the inner radius of nodal domains can be improved for quantum ergodic sequences of eigenfunctions, according to a certain power of the radius of shrinking balls on which the eigenfunctions equidistribute. We prove such improvements using a quick application of our recent results [He16], which give modified growth estimates for eigenfunctions that equidistribute on small balls. Since by [Han15, HeRi16] small scale QE holds for negatively curved manifolds on logarithmically shrinking balls, we get logarithmic improvements on the inner radius of eigenfunctions on such manifolds. We also get improvements for manifolds with ergodic geodesic flows. In addition using the small scale equidistribution results of [LeRu16], one gets polynomial betterments of [Ma08b] for toral eigenfunctions in dimensions $n \geq 3$. The results work only for a full density subsequence of eigenfunctions.

1. INTRODUCTION

Let (X, g) be a smooth compact connected boundaryless Riemannian manifold of dimension n . Suppose Δ_g is the positive Laplace-Beltrami operator on (X, g) and ψ_λ is a real-valued eigenfunction of Δ_g with eigenvalue $\lambda > 0$. Let Ω_λ be a nodal domain of ψ_λ and $\text{in}(\Omega_\lambda)$ be its inner radius. Mangoubi [Ma08a, Ma08b] has shown that ¹

$$(1.1) \quad a_1 \lambda^{-\frac{1}{2} - \frac{(n-1)(n-2)}{4n}} \leq \text{in}(\Omega_\lambda) \leq a_2 \lambda^{-\frac{1}{2}},$$

where a_1 and a_2 depend only on (X, g) . In this note we show that

Theorem 1.1. *There exists $r_0(g) > 0$ such that if $\lambda^{-1/2} < r_0(g)$, and if for some $r \in [\lambda^{-1/2}, r_0(g)]$ and for all geodesic balls $\{B_r(x)\}_{x \in X}$ we have*

$$(1.2) \quad K_1 r^n \leq \int_{B_r(x)} |\psi_\lambda|^2 dv_g \leq K_2 r^n,$$

for some positive constants K_1 and K_2 independent of x , then for $n \geq 3$

$$(1.3) \quad a_1 r^{-\frac{(n-1)(n-2)}{2n}} \lambda^{-\frac{1}{2} - \frac{(n-1)(n-2)}{4n}} \leq \text{in}(\Omega_\lambda)$$

¹In particular in dimension two, one has the optimal lower bound $a_1 \lambda^{-1/2}$.

A result of [HeRi16]², shows that on negatively curved manifolds, (1.2) holds for a full density subsequence with $r = (\log \lambda)^{-\kappa}$ for any $\kappa \in (0, \frac{1}{2n})$. Hence the following unconditional result on such manifolds is quickly obtained.

COROLLARY 1.2. Let (X, g) be a boundaryless compact connected smooth Riemannian manifold of dimension $n \geq 3$, with negative sectional curvatures. Let $\{\psi_{\lambda_j}\}_{j \in \mathbb{N}}$ be any ONB of $L^2(X)$ consisting of real-valued eigenfunctions of Δ_g with eigenvalues $\{\lambda_j\}_{j \in \mathbb{N}}$. Let $\varepsilon > 0$ be arbitrary. Then there exists $S_\varepsilon \subset \mathbb{N}$ of *full density*³ such that for $j \in S_\varepsilon$:

$$a_1 (\log \lambda_j)^{\frac{(n-1)(n-2)}{4n^2} - \varepsilon} \lambda_j^{-\frac{1}{2} - \frac{(n-1)(n-2)}{4n}} \leq \text{in}(\Omega_{\lambda_j}),$$

for some a_1 that only depends on (X, g) and ε .

One can also get the following improvements for quantum ergodic sequences of eigenfunctions. In fact it is enough to assume equidistribution on the configuration space X .

COROLLARY 1.3. Let (X, g) be a boundaryless compact connected smooth Riemannian manifold of dimension $n \geq 3$. Let $\{\psi_{\lambda_j}\}_{j \in S}$ be a sequence of real-valued eigenfunctions of Δ_g with eigenvalues $\{\lambda_j\}_{j \in S}$ such that for all $r \in (0, R_0)$, for some fixed $R_0 > 0$, and all $x \in X$

$$(1.4) \quad \int_{B_r(x)} |\psi_{\lambda_j}|^2 \rightarrow \frac{\text{Vol}_g(B_r(x))}{\text{Vol}_g(X)}, \quad \lambda_j \xrightarrow{j \in S} \infty.$$

Then along this sequence

$$\lim_{j \rightarrow \infty} \lambda_j^{\frac{1}{2} + \frac{(n-1)(n-2)}{4n}} \text{in}(\Omega_{\lambda_j}) = \infty.$$

In particular the above corollary holds for manifolds with ergodic geodesic flows by the quantum ergodicity theorem of [Sh74, CdV85, Ze87]. This means that given any ONB of eigenfunctions on such a manifold one can find a full density subsequence where (1.4), hence Corollary 1.3 holds.

We must also mention the work of Lester-Rudnick [LeRu16] where they proved that for a full density subsequence of toral eigenfunctions one has equidistribution at the shrinking rate $r = \lambda^{-\frac{1}{2n-2} + \epsilon}$. Of course, one can also use this and Theorem 1.1 to get improved lower bounds for such toral eigenfunctions.

COROLLARY 1.4. For any ONB $\{\psi_{\lambda_j}\}_{j \in \mathbb{N}}$ of real-valued eigenfunctions of Δ on the flat torus $\mathbb{T}^{n \geq 3}$, and any $\varepsilon > 0$, there exists a full density subset $S_\varepsilon \subset \mathbb{N}$ such that for $j \in S_\varepsilon$

$$a_1 \lambda_j^{-\frac{1}{2} - \frac{(n-2)^2}{4n} - \varepsilon} \leq \text{in}(\Omega_{\lambda_j}),$$

²In [Han15], this is proved for $\kappa \in (0, \frac{1}{2n})$.

³It means that $\lim_{N \rightarrow \infty} \frac{1}{N} \text{card}(S \cap [1, N]) = 1$.

where a_1 is a positive constant that depends only on n and ε .

1.1. Proofs of Theorem 1.1. The main idea is to use the modified growth estimates of our recent preprint [He16]. We recall that Donnelly-Fefferman [DoFe88] showed that an eigenfunction ψ_λ of Δ_g with eigenvalue λ satisfies

$$N(B_s(x)) := \log \left(\frac{\sup_{B_{2s}(x)} |\psi_\lambda|^2}{\sup_{B_s(x)} |\psi_\lambda|^2} \right) \leq c\sqrt{\lambda},$$

for all $s < s_0$ where s_0 and c depend only on (X, g) . In [He16], we have shown that under the assumption

$$K_1 r^n \leq \int_{B_r(x)} |\psi_j|^2 \leq K_2 r^n,$$

we have

$$(1.5) \quad N(B_s(x)) \leq cr\sqrt{\lambda}, \quad \text{for all } s < 10r,$$

where c is positive and is uniform in x , r , s , and λ , but depends on K_1 , K_2 , and (X, g) . We apply these growth estimates to the proof of [Ma08b].

We emphasize that in what follows $n \geq 3$. In [Ma08b], it is first proved that there exists $\varepsilon_0 \in (0, 1)$, sufficiently small and only dependent on (X, g) , such that for all embedded geodesic balls $B_R(p)$ with $\{\psi_\lambda = 0\} \cap B_{R/2}(p) \neq \emptyset$ we have

$$\frac{\text{Vol}(\{\psi_\lambda \geq 0\} \cap B_R(p))}{\text{Vol}(B_R(p))} \geq \frac{a_3}{N(\lambda)^{n-1}},$$

where

$$N(\lambda) = \sup_{x \in X, s \leq \varepsilon_0 \lambda^{-\frac{1}{2}}} N(B_s(x)),$$

and $a_3 > 0$ depends only on (X, g) . Then it is shown that

$$\text{in}(\Omega_\lambda) \geq a_4 \lambda^{-\frac{1}{2}} \inf_{B \in \mathcal{B}_\lambda} \left(\frac{\text{Vol}(\{\psi_\lambda \geq 0\} \cap B_R(p))}{\text{Vol}(B_R(p))} \right)^{\frac{n-2}{2n}},$$

where the infimum is taken over the set \mathcal{B}_λ of balls $B_R(p)$ such that $\{\psi_\lambda = 0\} \cap B_{R/2}(p) \neq \emptyset$, and $a_4 > 0$ depends only on (X, g) . Combining the last two inequalities, one obtains

$$\text{in}(\Omega_\lambda) \geq a_5 \lambda^{-\frac{1}{2}} N(\lambda)^{-\frac{(n-1)(n-2)}{2n}}.$$

Since $s \leq \varepsilon_0 \lambda^{-1/2} \leq r < 10r$, we can use our improved doubling estimates (1.5) to get $N(\lambda) \leq cr\sqrt{\lambda}$, which implies the theorem easily.

1.2. Proof of Corollary 1.3. This follows quickly from the following lemma together with Theorem 1.1.

LEMMA 1.5. Let $\{\psi_j\}_{j \in S}$ be a sequence of eigenfunctions of Δ_g with eigenvalues $\{\lambda_j\}_{j \in S}$ such that for some $R_0 > 0$, all $r \in (0, R_0)$, and all $x \in X$

$$(1.6) \quad \int_{B_r(x)} |\psi_j|^2 \rightarrow \frac{\text{Vol}_g(B_r(x))}{\text{Vol}_g(X)}, \quad \lambda_j \xrightarrow{j \in S} \infty.$$

Then there exists $r_0(g)$ such that for each $r \in (0, r_0(g))$ there exists Λ_r such that for $\lambda_j \geq \Lambda_r$ we have

$$K_1 r^n \leq \int_{B_r(x)} |\psi_j|^2 \leq K_2 r^n,$$

uniformly for all $x \in X$. Here, K_1 and K_2 are independent of r , j , and x .

Proof. This lemma is obvious when x is fixed but to show that it holds uniformly in x one can use a covering argument as presented in [He16]. \square

Remark 1.6. It is clear from Theorem 1.1 that if (1.2) holds for $r = \lambda^{-\frac{1}{2}+\varepsilon}$ for arbitrary $\varepsilon > 0$, then we have

$$a_1(\varepsilon) \lambda^{-\frac{1}{2}-\varepsilon} \leq \text{in}(\Omega_\lambda) \leq a_2 \lambda^{-\frac{1}{2}},$$

for any $\varepsilon > 0$. It is a natural, but seemingly very difficult to prove, conjecture that $r = \lambda^{-\frac{1}{2}+\varepsilon}$ is the optimal rate for the eigenfunctions of negatively curved manifolds. A result of [LeRu16] shows that this optimal rate of shrinking is satisfied for a full density subsequence of any ONB of eigenfunctions of the flat 2-torus \mathbb{T}^2 .

Remark 1.7. We have used both local and global harmonic analysis of eigenfunctions. The local analysis is contained in the work of Mangoubi, where lower bounds for the inner radius in terms of the doubling index are given. The global analysis is contained in the small scale QE results of [Han15, HeRi16] and also the QE results of [Sh74, CdV85, Ze87], where the global behavior of the geodesic flow and the wave operator are considered to obtain equidistribution on small balls. Hence, any non-optimal results that give improvements on the results of [Ma08b] and is only based on the local analysis of eigenfunctions can be improved using our hybrid method.

Remark 1.8. The upper bound in (1.1) is an immediate consequence of a result of Brüning [Br78], which says that there exists $a_2 > 0$ dependent only on (X, g) such that every geodesic ball of radius $a_2 \lambda^{-1/2}$ contains a zero of ψ_λ . See [Ze08] for a simple proof using the domain monotonicity property of Dirichlet eigenvalues.

Remark 1.9. The paper [GeMa16] gives a refinement of the result of Mangoubi using Brownian motions techniques. To be precise, the authors show that the inscribed ball that gives the lower bound of [Ma08b] can be centered at the maximum of the eigenfunctions ψ_λ on its nodal domain Ω_λ .

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